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Spinning Particles in NUT–Reissner–Nordstrom Space–Time

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We study the geodesic motion of pseudo-classical spinning particles in the NUT–Reissner–Nordstrom space–time. We investigate the generalized Killing equations for spinning space and derive the constants of the motion in terms of the solutions of these equations. We give an analysis of the motion on a cone and on a plane.

1. INTRODUCTION

In recent years there has been an interest in studying spinning particles, such as Dirac fermions, in curved space–times. These spinning particles are described by pseudo-classical mechanics models in which the spin degrees of freedom are characterized in terms of anticommuting Grassmann variables (Barducci *et al.*, 1976; Berezin and Marinov, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Gibbons *et al.*, 1993; Rietdijk and van Holten, 1990, 1993; van Holten and Rietdijk, 1993). Rietdijk and van Holten (1993) studied spinning particles in the Schwarzschild space–time. Visinescu (1994a,b), Vaman and Visinescu (1996, 1998, 1999), van Holten (1995), and Baleanu (1994) investigated pseudo-classical spinning particles in the Taub–NUT space–time. In a previous study (Ali and Ahmed, 2000), we studied spinning particles in the Reissner–Nordstrom (RN) space–time. In the present paper we investigate the geodesic motion of pseudo-classical spinning particles in the NUT–RN space–time, which is the RN space–time generalized with NUT (or magnetic mass) parameter. This work may be interesting in that it provides results parallel to those obtained in the RN and Taub–NUT space–times.

The NUT–RN space–time includes the NUT space–time that is sometimes considered as unphysical. According to Misner (1963), the NUT space–time is one that does not admit an interpretation without a periodic time coordinate, a space–time without reasonable spacelike surfaces, and an asymptotically zero curvature

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space–time that apparently does not admit asymptotically rectangular coordinates. McGuire and Ruffini (1975) suggested that the space–times endowed with NUT parameter should never be directly, physically interpreted. This makes the study of pseudo-classical spinning particles in the NUT–RN space–time interesting.

The plan of this paper is as follows. In section 2 we summarize the relevant equations for the motion of spinning particles in curved space–time. The generalized Killing equations for spinning spaces are investigated, and the constants of motion are derived in terms of the solutions of these equations. In section 3 we analyze the motion of pseudo-classical spinning particles in the NUT–RN space–time. We examine the generalized Killing equations for this space–time and describe the derivation of the constants of motion. In section 4 we solve the equations derived in the previous section for the special case of motion on a cone and in a plane for which we choose $\theta = \pi/2$. In section 5 we investigate a new type of supersymmetry in the NUT–RN space–time. This *nongeneric* supersymmetry is generated by the Killing–Yano tensor. In section 6 we present our remarks.

2. MOTION IN SPINNING SPACE

The geodesic for spinning space can be obtained from the action,

$$S = \int_a^b d\tau \left(\frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x)\psi^\mu \frac{D\psi^\nu}{D\tau} \right) \tag{1}$$

The overdot, here and in the following, denotes an ordinary derivative with respect to proper time, $d/d\tau$. The covariant derivative of Grassmann coordinates ψ^μ is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\nu} \psi^\nu \tag{2}$$

The trajectories along which the action is stationary under arbitrary variations δx^μ and $\delta\psi^\mu$ vanishing at the endpoints are given by

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\, \dot{x}^\lambda \dot{x}^\nu = \frac{1}{2i} \psi^\kappa \psi^\lambda R^\mu_{\kappa\lambda\nu} \dot{x}^\nu \tag{3}$$

$$\frac{D\psi^\mu}{D\tau} = 0 \tag{4}$$

The antisymmetric tensor

$$S^{\mu\nu} = -i\psi^\mu \psi^\nu \tag{5}$$

can formally be regarded as the spin-polarization tensor of the particle [Barducci *et al.*, 1976; Berezin and Marinov, 1977; Brink *et al.*, 1976, 1977; Casalbuoni, 1976; Rietdijk and van Holten, 1990, 1993; van Holten and Rietdijk, 1993]. The equations of motion can be expressed in terms of this tensor. Equation (4) asserts that the spin is covariantly constant,

$$\frac{DS^{\mu\nu}}{D\tau} = 0 \tag{6}$$

Equation (5) implies the existence of a spin-dependent gravitational force

$$\frac{D^2 x^\mu}{D\tau^2} = \frac{1}{2} S^{\kappa\lambda} R^{\mu}_{\kappa\lambda\,\nu} \dot{x}^\nu \tag{7}$$

The spacelike components S^{ij} are proportional to the particle's magnetic dipole moment, while the timelike components S^{io} represent the electric dipole moment. In the rest frame, the components S^{io} vanish for free Dirac particles like free electrons and quarks. This leads to the covariant constraint.

$$g_{\nu\lambda}(x) S^{\mu\nu} \dot{x}^\lambda = 0 \tag{8}$$

In Grassmann coordinates it takes the form

$$g_{\mu\nu}(x) \dot{x}^\mu \psi^\nu = 0 \tag{9}$$

The concept of a Killing vector can be generalized to the case of spinning manifolds. For this purpose it is necessary to consider variations of x^μ and ψ^μ that leave the action (1) invariant modulo boundary terms. Let us take the variations to be of the form

$$\delta x^\mu = \mathcal{R}^\mu(x, \dot{x}, \psi) = R^{(1)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1 \cdots \nu_n}(x, \psi),$$

$$\delta \psi^\mu = \mathcal{S}^\mu(x, \dot{x}, \psi) = S^{(0)\mu}\,(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} S^{(n)\mu}_{\nu_1 \cdots \nu_n}(x, \psi) \tag{10}$$

The Lagrangian transforms into a total derivative

$$\delta S = \int_a^b d\tau\, \frac{d}{d\tau} \left(\delta x^\mu p_\mu - \frac{i}{2} \delta \psi^\mu g_{\mu\nu} \psi^\nu - \mathcal{J}(x, \dot{x}, \psi) \right) \tag{11}$$

where p_μ is the canonical momentum conjugate to x^μ

$$p_\mu = g_{\mu\nu} \dot{x}^\nu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu = \Pi_\mu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu \tag{12}$$

Π_μ being the covariant momentum. If the equations of motion are satisfied, it follows from Noether's theorem that the quantity $\mathcal{J}$ is a constant of motion.

We consider the world-line Hamiltonian given by

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu \tag{13}$$

For any constant of motion $\mathcal{J}(x, \Pi, \psi)$, the bracket with H vanishes

$$\{H, \mathcal{J}\} = 0 \tag{14}$$

where the Poisson–Dirac brackets for two functions of the covariant phase–space variables (x, Π, ψ) is defined by

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} \mathcal{D}_\mu G - \mathcal{R}_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^\mu} \frac{\partial G}{\partial \psi_\mu} \tag{15}$$

with

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma^\lambda_{\mu\nu} \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} - \Gamma^\lambda_{\mu\nu} \psi^\nu \frac{\partial F}{\partial \psi^\lambda}$$

$$\mathcal{R}_{\mu\nu} = \frac{i}{2} \psi^\sigma \psi^\lambda R_{\sigma\lambda\mu\nu} \tag{16}$$

Here, a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F =$ (even, odd).

If we expand $\mathcal{J}(x, \Pi, \psi)$ in a power series in the covariant momentum

$$\mathcal{J} = \mathcal{J}^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \Pi^{\mu_1} \cdots \Pi^{\mu_n} \mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi) \tag{17}$$

then the bracket $\{H, \mathcal{J}\}$ vanishes for arbitrary Π_μ if and only if the components of $\mathcal{J}$ satisfy the generalized Killing equations (Rietdijk and van Holten, 1990; van Holten and Rietdijk, 1993)

$$D_{(\mu_{n+1}} \mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n)} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1 \cdots \mu_n}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu_{n+1})_\lambda} \psi^\lambda = \frac{i}{2} \psi^\sigma \psi^\lambda R_{\sigma\lambda\nu(\mu_{n+1}} \mathcal{J}^{(n+1)^\nu}_{\mu_1 \cdots \mu_n)} \tag{18}$$

where the parentheses denote full symmetrization over the indices enclosed.

In general, the symmetries of a spinning particle model can be divided into two classes. First, there are four independent *generic* symmetries, which exist in any theory [Rietdijk and van Holten, 1990; van Holten and Rietdijk, 1993]:

(i) Proper-time translations generated by the Hamiltonian H (13);
(ii) Supersymmetry generated by the supercharge

$$Q = \prod_\mu \psi^\mu \tag{19}$$

(iii) Chiral symmetry generated by the chiral charge

$$\Gamma_* = \frac{i^{[d/2]}}{d!} \sqrt{g} \varepsilon_{\mu_1 \cdots \mu_d} \psi^{\mu_1} \cdots \psi^{\mu_d} \tag{20}$$

(iv) Dual supersymmetry, generated by the dual supercharge

$$Q^* = i\{\Gamma_*, Q\} = \frac{i^{[d/2]}}{(d-1)!}\sqrt{g}\,\varepsilon_{\mu_1\cdots\mu_d}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d} \tag{21}$$

where d is the dimension of space–time.

As a rule one has the freedom to choose the value of the supercharge Q and any choice gives a consistent model. The condition for the absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) as formulated in Eq. (9) implies

$$Q = 0 \tag{22}$$

However, for the time being, we shall not fix the value of the supercharge to keep the presentation as general as possible.

The second kind of conserved quantities, called *nongeneric*, depend on the explicit form of the metric $g_{\mu\nu}(x)$. In recent literature there have been exhibited the constants of motion in the schwarzschild [Rietdijk and van Holten, 1993], Taub–NUT [Vaman and Visinescu, 1996; 1998; van Holten 1995; Visinescu, 1997, 1994a,b], Kerr-Newman [Gibbons *et al.*, 1993; van Holten, 1994] spinning spaces. In what follows we shall deal with the *nongeneric* constants of motion in connection with the Killing equation (18).

We remind that a tensor $f_{\mu\nu}$ is called a Killing–Yano tensor of valence 2 [Dietz and Rudinger, 1981; Yano, 1952] if it is completely antisymmetric and it satisfies the equation

$$D_\nu f_{\lambda\mu} + D_\lambda f_{\nu\mu} = 0 \tag{23}$$

The Killing–Yano tensors play a key role in the Dirac theory on a curved space–time [Carter and McLenagham, 1970]. The study of the generalized Killing equations strengthens the connection of the Killing–Yano tensors with the supersymmetric classical and quanntum mechanics of curved manifolds.

The *nongeneric* symmetry of the theory is generated by the phase–space function Q_f,

$$Q_f = \mathcal{J}^{(1)\mu}\Pi_\mu + \mathcal{J}^{(0)} \tag{24}$$

where $\mathcal{J}^{(1)}(x, \psi)$ and $\mathcal{J}^{(0)}(x, \psi)$ are independent of Π. This charge generates the supersymmetry transformation

$$\delta x^\mu = -i\varepsilon f^\mu_a \psi^a \equiv -i\varepsilon \mathcal{J}^{(1)\mu} \tag{25}$$

where the infinitesimal parameter ε of the transformation is Grassmann odd. Greek and latin indices, which refer to world and Lorentz indices respectively, are converted into each other by the vielbein (tetrad) $e^a_\mu(x)$ and its inverse $e^\mu_a(x)$. When the ansatz (24) is inserted into the generalized Killing equations (18), it follows that

$$\mathcal{J}^{(0)} = \frac{i}{3!} C_{abc}(x) \psi^a \psi^b \psi^c \tag{26}$$

where the tensors f_a^μ and C_{abc} satisfy the conditions

$$D_\mu f_{\nu a} + D_\nu f_{\mu a} = 0 \tag{27}$$

and

$$D_\mu C_{abc} = -\left(R_{\mu\nu ab} f_c^\nu + R_{\mu\nu bc} f_a^\nu + R_{\mu\nu ca} f_b^\nu\right) \tag{28}$$

Using Eqs. (23) and (28) the tensor C_{abc} can be expressed as follows:

$$C_{abc} = -2\ D_{[a} f_{bc]} = -2 e_a^\mu e_b^\nu e_c^\lambda\ D_{[\mu} f_{\nu\lambda]} \tag{29}$$

where the square brackets denote full antisymmetrization over the indices enclosed. Let there be N such symmetries specified by N sets of tensors (f_{ia}^μ, C_{iabc}), $i = 1, \ldots, N$. The corresponding generators will be

$$Q_i = f_{ia}^\mu \Pi_\mu \psi^a + \frac{i}{3!}\ C_{iabc}\ \psi^a \psi^b \psi^c \tag{30}$$

Obviously, for $f_a^\mu = e_a^\mu$ and $C_{abc} = 0$, the supercharge (19) is precisely of this form. It is therefore convenient to assign the index $i = 0$: $Q = Q_0$, $e_a^\mu = f_{0a}^\mu$, etc., when we refer to the quantities defining the standard supersymmetry.

The covariant form (15) of poisson–Dirac brackets gives the following algebra for the conserved charges Q_i:

$$\{Q_i, Q_j\} = -2i\ Z_{ij} \tag{31}$$

where

$$Z_{ij} = \frac{1}{2}\ K_{ij}^{\mu\nu}\ \Pi_\mu \Pi_\nu + I_{ij}^\mu \Pi_\mu + G_{ij} \tag{32}$$

and

$$K_{ij}^{\mu\nu} = \frac{1}{2}\left(f_{ia}^\mu f_j^{\nu a} + f_{ia}^\nu f_j^{\nu a}\right) \tag{33}$$

$$\begin{aligned} I_{ij}^\mu &= \frac{i}{2}\psi^a \psi^b I_{ijab}^\mu \\ &= \frac{i}{2}\psi^a \psi^b \left(f_{ib}^\nu D_\nu f_{ja}^\mu + f_{jb}^\nu D_\nu f_{ia}^\mu + \frac{1}{2} f_i^{\mu c} C_{jabc} + \frac{1}{2} f_j^{\mu c} C_{iabc} \right) \end{aligned} \tag{34}$$

$$\begin{aligned} G_{ij} &= -\frac{1}{4}\psi^a \psi^b \psi^c \psi^d\ G_{ijabcd} \\ &= -\frac{1}{4}\psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f_{ic}^\mu f_{jd}^\nu + \frac{1}{2} C_{iab}^e\ C_{jcde} \right) \end{aligned} \tag{35}$$

We note that $K_{ij\mu\nu}$ is a symmetric Killing tensor of second rank:

$$D_{(\lambda}K_{ij\mu\nu)} = 0 \tag{36}$$

I^{μ}_{ij} is the corresponding Killing vector:

$$\mathscr{D}_{(\mu}I_{ij\nu)} = -\frac{i}{2}\psi^a\psi^b\ D_{(\mu}I_{ij\nu)ab} = \frac{i}{2}\psi^a\psi^b R_{ab\lambda(\mu}\ K^{\lambda}_{ij\nu)} \tag{37}$$

and G_{ij} is the corresponding Killing scalar:

$$\mathscr{D}_{\mu}G_{ij} = -\frac{1}{4}\psi^a\psi^b\psi^c\psi^d\ D_{\mu}G_{ijabcd} = \frac{i}{2}\psi^a\psi^b R_{ab\lambda\mu}\ I^{\lambda}_{ij} \tag{38}$$

The functions Z_{ij} satisfy the generalized Killing equations. Hence their bracket with the Hamiltonian vanishes and they are constants of motion. For $i = j = 0$, Eq. (30) gives the usual supersymmetry algebra

$$\{Q, Q\} = -2iH \tag{39}$$

If i or j is not equal to zero, Z_{ij} correspond to new bosonic symmetries unless $K^{\mu\nu}_{ij} = \lambda_{(ij)}g^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (may be zero). Then the corresponding Killing vector I^{μ}_{ij} and Killing scalar G_{ij} disappear identically. Further, if $\lambda_{(ij)} \neq 0$ the corresponding supercharges close on the Hamiltonian and hence there exists a second supersymmetry of the standard type. Thus the theory admits an N-extended supersymmetry with $N \geq 2$. Again, if we have a second independent Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, there exists a genuine new type of supersymmetry.

The quantity Q_i is a superinvariant

$$\{Q_i, Q\} = 0 \tag{40}$$

for the bracket defined by (15). The condition for this is

$$K^{\mu\nu}_{oi} = f^{\mu}_a e^{\nu a} + f^{\nu}_a e^{\mu a} = 0 \tag{41}$$

As the Z_{ij} are symmetric in (ij) we can diagonalize them. This provides the algebra

$$\{Q_i, Q_j\} = -2i\delta_{ij}\ Z_i \tag{42}$$

where Z_i are $N + 1$ conserved bosonic charges. If all Q_i satisfy condition (41), the first of these diagonal charges (with $i = 0$) is the Hamiltonian: $Z_0 = H$.

3. GEODESIC MOTION IN NUT–REISSNER–NORDSTROM SPACE–TIME

In this section we shall apply the results of the previous section to investigate the geodesic motion of a spinning particle in the NUT–RN space–time described by the matric

$$ds^2 = -N(r)\left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\varphi\right]^2 + N^{-1}(r)\,dr^2$$
$$+ (r^2+n^2)[d\theta^2 + \sin^2\theta\, d\varphi^2] \qquad (43)$$

where

$$N(r) = 1 - \frac{2}{r^2+n^2}\left[Mr + n^2 - \frac{e^2}{2}\right], \qquad (44)$$

n is the NUT (or magnetic mass) parameter, e the charge, and M the total mass of the gravitating body. The space–time, given by (43) and (44), gives (i) the NUT space–time for $e = 0$, (ii) the Reissner–Nordstrom space–time for $n = 0$, and (iii) the Schwarzschild space–time for $n = e = 0$.

Spaces with a metric of the form given above have an isometry group $SU(2) \times U(1)$. The metric has the four Killing vectors:

$$D^{(\alpha)} = R^{(\alpha)\mu}\partial_\mu, \quad \alpha = 0, \ldots, 3, \qquad (45a)$$

where

$$D^{(0)} = \frac{\partial}{\partial t}, \qquad D^{(1)} = -\sin\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi} - 2n\tan\left(\frac{\theta}{2}\right)\cos\varphi\,\frac{\partial}{\partial t}$$
$$D^{(2)} = \cos\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi} - 2n\tan\left(\frac{\theta}{2}\right)\sin\varphi\,\frac{\partial}{\partial t},$$
$$D^{(3)} = \frac{\partial}{\partial\varphi} - 2n\,\frac{\partial}{\partial t} \qquad (45b)$$

$D^{(0)}$, which generates the $U(1)$ of t translations, commutes with the other Killing vectors. The remaining three vectors obey an $SU(2)$ algebra with

$$\left[D^{(a)}, D^{(b)}\right] = -\varepsilon^{abc}D^{(c)}, \quad (a,\ b,\ c = 1, 2, 3) \qquad (46)$$

This can be contrasted with the Reissner–Nordstrom space–time, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic mass (Mueller and Perry, 1986; Sorkin, 1983).

In the purely bosonic case these invariances would correspond to conservation of the so-called "relative electric charge" and the angular momentum (Cordani *et al.*, 1988; Feher and Horvathy, 1987; Gibbons and Manton, 1986; Gibbons and Ruback, 1987, 1988; Visinescu, 1993; Zimmerman and Shahir, 1989).

$$q = -N\left(\dot{t} + 4n\sin^2\left(\frac{\theta}{2}\right)\dot{\varphi}\right) \qquad (47)$$

$$\boldsymbol{j} = \boldsymbol{r} \times \boldsymbol{p} + 2nq\frac{\boldsymbol{r}}{r} \qquad (48)$$

The first generalized Killing equation has the form

$$B^{(\alpha)}_{,\mu} + \frac{\partial B^{(\alpha)}}{\partial \psi^{\sigma}}\, \Gamma^{\sigma}_{\mu\lambda} \psi^{\lambda} = \frac{i}{2} \psi^{\lambda} \psi^{\sigma}\; R_{\lambda\sigma\nu\mu} R^{(\alpha)\nu} \tag{49}$$

This shows that for each Killing vector, $R^{(\alpha)}_{\mu}$, there is an associated Killing scalar, $B^{(\alpha)}$. So, if we limit ourselves to variations (10) that terminate after the terms linear in $\dot{x}^{\mu}$, we get the constants of motion

$$J^{(\alpha)} = B^{(\alpha)} + \dot{x}^{\mu} R^{(\alpha)}_{\mu} \tag{50}$$

Equation (50) asserts that the Killing scalars contribute to the "relative electric charge" and the total angular momentum.

Inserting the expressions for the connection and the Riemann curvature components corresponding to the NUT–RN space–time in (49), we obtain for the Killing scalars

$$B^{(0)} = V S^{tr} - 4nV\; \sin^2 \left(\frac{\theta}{2}\right) S^{r\varphi} - 2nN\; \sin\theta\, S^{\theta\varphi} \tag{51a}$$

$$\begin{aligned} B^{(1)} = {} & -2nV\; \cos\varphi\; \tan\left(\frac{\theta}{2}\right) (1 + \cos\theta)\, S^{tr} \\ & - nN\; \cos\varphi\; \cos\theta\, S^{t\theta} - \; r\; \sin\varphi\, S^{r\theta} \\ & + \cos\varphi\; \cot\theta \left[8n^2 V\; \cos\theta\; \tan\left(\frac{\theta}{2}\right)\; \sin^2\left(\frac{\theta}{2}\right) - r\; \sin^2\theta \right] S^{r\varphi} \\ & + \cos\varphi \left[(r^2 + n^2) \sin^2\theta + 4n^2 N - 8n^2 N \tan^2\left(\frac{\theta}{2}\right) \right] S^{\theta\varphi} \end{aligned} \tag{51b}$$

$$\begin{aligned} B^{(2)} = {} & -2nV \sin\varphi \tan\left(\frac{\theta}{2}\right) (1 + \cos\theta) S^{tr} - nN \sin\varphi \cos\theta S^{t\theta} + r \cos\varphi S^{r\theta} \\ & + \sin\varphi \cot\theta \left[8n^2 V \cos\theta \tan\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\theta}{2}\right) - r \sin^2\theta \right] S^{r\varphi} \\ & + \sin\varphi \left[(r^2 + n^2) \sin^2\theta + 4n^2 N - 8n^2 N \tan^2\left(\frac{\theta}{2}\right) \right] S^{\theta\varphi} \end{aligned} \tag{51c}$$

$$\begin{aligned} B^{(3)} = {} & -2nV \cos\theta S^{tr} + 4nV \sin^2\left(\frac{\theta}{2}\right) S^{t\theta} \\ & + \left[r \sin^2\theta + 8n^2 V \sin^2\left(\frac{\theta}{2}\right) \cos^2\theta \right] S^{r\varphi} \\ & + \sin\theta \left[(r^2 + n^2) \cos\theta - 2n^2 N \left(1 + 4 \sin^2\left(\frac{\theta}{2}\right) \right) \right] S^{\theta\varphi} \end{aligned} \tag{51d}$$

where

$$V = \frac{1}{(r^2+n^2)^2}[M(r^2-n^2)+(2n^2-e^2)r] \tag{51e}$$

and N is given by (44).

Taking into account the contribution of the Killing scalars, one finds for the conserved quantities $\mathcal{J}^{(\alpha)}$,

$$\mathcal{J}^{(0)} = B^{(0)} + q \tag{52a}$$

$$\mathcal{J}^{(1)} = B^{(1)} - (r^2+n^2)\sin\varphi\,\dot{\theta} - (r^2+n^2)\cos\theta\sin\theta\cos\varphi\,\dot{\varphi} + 2nq\sin\theta\cos\varphi \tag{52b}$$

$$\mathcal{J}^{(2)} = B^{(2)} + (r^2+n^2)\cos\varphi\,\dot{\theta} - (r^2+n^2)\cos\theta\sin\theta\sin\varphi\,\dot{\varphi} + 2nq\sin\theta\sin\varphi \tag{52c}$$

$$\mathcal{J}^{(3)} = B^{(3)} + (r^2+n^2)\sin^2\theta\,\dot{\varphi} + 2nq\cos\theta \tag{52d}$$

It is obvious that the “relative electric charge,” q, is no longer conserved, contrasting with the purely bosonic case. Also, the conserved total angular momentum is the sum of the orbital angular momentum, the Poincare contribution and the spin angular momentum

$$\boldsymbol{\mathcal{J}} = \boldsymbol{B} + \boldsymbol{j} \tag{53}$$

with $\boldsymbol{\mathcal{J}} = (\mathcal{J}^{(1)}, \mathcal{J}^{(2)}, \mathcal{J}^{(3)})$ and $\boldsymbol{B} = (B^{(1)}, B^{(2)}, B^{(3)})$.

From (52) we can derive two very interesting relations

$$\mathcal{J}^{(1)}\sin\varphi - \mathcal{J}^{(2)}\cos\varphi = -rS^{r\theta} - (r^2+n^2)\dot{\theta} \tag{54}$$

$$\begin{aligned}
&\mathcal{J}^{(1)}\sin\theta\cos\varphi + \mathcal{J}^{(2)}\sin\theta\sin\varphi + \mathcal{J}^{(3)}\cos\theta \\
&\quad = 2n\mathcal{J}^{(0)} - 4nVS^{tr} + \left(4V\sin^2\left(\frac{\theta}{2}\right) - N\sin\theta\right)n\cos\theta S^{t\theta} \\
&\qquad + 8n^2V\sin^2\left(\frac{\theta}{2}\right)\left[1+\cos^2\theta\left(\cos\theta+\tan\left(\frac{\theta}{2}\right)\right)\right]S^{r\varphi} + \left[(r^2+n^2)\right. \\
&\qquad \left. + 8n^2N\left(1-\tan^2\left(\frac{\theta}{2}\right)\right) - 2n^2N\left(1+4\sin^2\left(\frac{\theta}{2}\right)\right)\cos\theta\right]\sin\theta S^{\theta\varphi}
\end{aligned} \tag{55}$$

Equation (55) expresses the fact that the total angular momentum in the radial direction receives contributions from the spin angular momentum, the orbital angular momentum in that direction being zero.

In addition there are four universal conserved charges described in the previous section. Using the notation from this section, we have

(i) The energy

$$E = \frac{1}{2N}\dot{r}^2 + \frac{1}{2}(r^2+n^2)(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - \frac{1}{2}N\left(\dot{t} + 4n\sin^2\left(\frac{\theta}{2}\right)\dot{\varphi}\right)^2 \tag{56}$$

(ii) The supercharge

$$Q = \frac{1}{N}\dot{r}\psi^r + (r^2+n^2)\dot{\theta}\psi^\theta + q\psi^t + \left[4n\sin^2\left(\frac{\theta}{2}\right)q + (r^2+n^2)\sin^2\theta\dot{\varphi}\right]\psi^\varphi \tag{57}$$

(iii) The chiral charge

$$\Gamma_* = (r^2+n^2)\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^t \tag{58}$$

(iv) The dual supercharge

$$Q^* = (r^2+n^2)\sin\theta(\dot{r}\psi^\theta\psi^\varphi\psi^t - \dot{\theta}\psi^r\psi^\varphi\psi^t + \dot{\varphi}\psi^r\psi^\theta\psi^t - \dot{t}\psi^r\psi^\theta\psi^\varphi) \tag{59}$$

Finally, keeping in mind that ψ^μ is covariantly constant as formulated in (4), the rate of change of spin is

$$\dot{\psi}^r = [rN - (r^2+n^2)V](\dot{\theta}\psi^\theta + \sin^2\theta\dot{\varphi}\psi^\varphi) \tag{60a}$$

$$\dot{\psi}^\theta = -\frac{r\dot{\theta}}{r^2+n^2}\psi^r - \frac{r\dot{r}}{r^2+n^2}\psi^\theta + \sin\theta\left[\left(\cos\theta - \frac{4n^2N}{r^2+n^2}\sin^2\left(\frac{\theta}{2}\right)\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]\psi^\varphi \tag{60b}$$

$$\dot{\psi}^\varphi = \frac{nN}{r^2+n^2}\mathrm{cosec}\,\theta\,\dot{\theta}\psi^t - \frac{r\dot{\varphi}}{r^2+n^2}\psi^r - \left(\cot\theta\,\dot{\varphi} + \frac{nq}{r^2+n^2}\mathrm{cosec}\,\theta\right)\psi^\theta - \left[\frac{r\dot{r}}{r^2+n^2} + \left(\cot\theta - \frac{2n^2N}{r^2+n^2}\tan\left(\frac{\theta}{2}\right)\right)\dot{\theta}\right]\psi^\varphi \tag{60c}$$

$$\dot{\psi}^t = \left(\frac{V}{N}\dot{r} - \frac{2n^2N}{r^2+n^2}\tan\frac{\theta}{2}\dot{\theta}\right)\psi^t + \left[4n\sin^2\frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{2V}{N}\right)\dot{\varphi} - \frac{V}{N^2}q\right]\psi^r - \left(2n\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\dot{\varphi} - \frac{2n^2q}{r^2+n^2}\tan\frac{\theta}{2}\right)\psi^\theta + \left[4n\sin^2\frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{V}{N}\right)\dot{r} - 2n\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\left(1 + \frac{4n^2N}{r^2+n^2}\right)\dot{\theta}\right]\psi^\varphi \tag{60d}$$

As a rule these complicated equations could be integrated to obtain the full solution of the equations of motion for the usual coordinates and Grassmann coordinates. These equations are quite intricate and the general solution is by no means illuminating. Instead of the general solution, we shall discuss special solutions in the next section for the motion on a cone and in a plane.

4. SPECIAL SOLUTIONS

In this section we solve the equations derived in the previous section for the motion on a cone and in a plane. We first consider the motion on a cone.

Let us choose the z axis along $\boldsymbol{J}$ so that the motion of the particle may be conveniently described in terms of polar coordinates

$$\boldsymbol{r} = r\boldsymbol{e}(\theta, \varphi) \tag{61}$$

with

$$\boldsymbol{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{62}$$

The equations of motion for the spin components when $\dot{\theta} = 0$ are

$$\begin{aligned}\dot{S}^{r\theta} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\theta} + \sin\theta\left[\left(\cos\theta - \frac{4n^2N}{r^2+n^2}\sin^2\left(\frac{\theta}{2}\right)\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]S^{r\varphi}\\ &\quad - [rN - (r^2+n^2)V]\sin^2\theta\dot{\varphi}S^{\theta\varphi}\end{aligned} \tag{63a}$$

$$\dot{S}^{r\varphi} = -\frac{r\dot{r}}{r^2+n^2}S^{r\varphi} - \left(\cot\theta\dot{\varphi} + \frac{nq}{r^2+n^2}\operatorname{cosec}\theta\right)S^{r\theta} \tag{63b}$$

$$\dot{S}^{\theta\varphi} = -\frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} + \frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta} \tag{63c}$$

$$\begin{aligned}\dot{S}^{\theta t} &= \left(\frac{V}{N} - \frac{r}{r^2+n^2}\right)\dot{r}S^{\theta t}\\ &\quad + \sin\theta\left[\left(\cos\theta - \frac{4n^2N}{r^2+n^2}\sin^2\left(\frac{\theta}{2}\right)\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]S^{\varphi t}\\ &\quad - \left[4n\sin^2\left(\frac{\theta}{2}\right)\left(\frac{r}{r^2+n^2} - \frac{2V}{N}\right)\dot{\varphi} - \frac{V}{N^2}q\right]S^{r\theta}\\ &\quad + 4n\sin^2\left(\frac{\theta}{2}\right)\left(\frac{r}{r^2+n^2} - \frac{V}{N}\right)\dot{r}S^{\theta\varphi}\end{aligned} \tag{63d}$$

$$\dot{S}^{rt} = [rN - (r^2+n^2)V]\sin^2\theta\dot{\varphi}S^{\varphi t} + \frac{V}{N}\dot{r}S^{rt}$$
$$-2n\tan\left(\frac{\theta}{2}\right)\left(\sin^2\left(\frac{\theta}{2}\right)\dot{\varphi} - \frac{nq}{r^2+n^2}\right)S^{r\theta}$$
$$+4n\sin^2\left(\frac{\theta}{2}\right)\left(\frac{r}{r^2+n^2} - \frac{V}{N}\right)\dot{r}S^{r\varphi} \quad (63e)$$

$$\dot{S}^{\varphi t} = -\frac{r\dot{\varphi}}{r^2+n^2}S^{rt} - \operatorname{cosec}\theta\left(\cos\theta\dot{\varphi} + \frac{nq}{r^2+n^2}\right)S^{\theta t} + \left(\frac{V}{N} - \frac{r}{r^2+n^2}\right)\dot{r}S^{\varphi t}$$
$$-\left[4n\sin^2\left(\frac{\theta}{2}\right)\left(\frac{r}{r^2+n^2} - \frac{2V}{N}\right)\dot{\varphi} - \frac{V}{N^2}q\right]S^{r\varphi}$$
$$+2n\tan\left(\frac{\theta}{2}\right)\left(\sin^2\left(\frac{\theta}{2}\right)\dot{\varphi} - \frac{nq}{r^2+n^2}\right)S^{\theta\varphi} \quad (63f)$$

Since we are looking for solutions with $\dot{\theta} = 0$ and because $\mathcal{J}^{(1)} = \mathcal{J}^{(2)} = 0$, we have from (54),

$$S^{r\theta} = 0 \quad (64)$$

This relation implies $\Gamma_* = 0$.

A particular solution may be obtained, if we choose $S^{\varphi t} = 0$, in the from

$$S^{r\varphi} = \frac{\mathrm{C}^{r\varphi}}{\sqrt{r^2+n^2}} \quad (65a)$$

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2} \quad (65b)$$

$$S^{\theta t} = \left(\frac{N}{r^2+n^2}\right)^{1/2}\mathrm{C}^{\theta t} - 2n\sin^2\left(\frac{\theta}{2}\right)\frac{\mathrm{C}^{\theta\varphi}}{r^2+n^2} \quad (65c)$$

$$S^{rt} = \sqrt{N}C^{rt} - 4n\sin^2\left(\frac{\theta}{2}\right)\frac{C^{r\varphi}}{\sqrt{r^2+n^2}} \quad (65d)$$

where $C^{r\varphi}$, $C^{\theta\varphi}$, $C^{\theta t}$, C^{rt} are Grassmann constants.

We investigate the case in which $Q = 0$ (Eq. (22)). From Rietdijk and van Holten (1993), we have $\Gamma_* = Q^* = 0$. For the spin components we deduce the following relations:

$$\frac{\dot{r}}{N}S^{r\theta} = \left[(r^2+n^2)\sin^2\theta\,\dot{\varphi} + 4n\sin^2\left(\frac{\theta}{2}\right)q\right]S^{\theta\varphi} + qS^{\theta t} \quad (66a)$$

$$\frac{\dot{r}}{N}S^{r\varphi} = qS^{\varphi t} \tag{66b}$$

$$\frac{\dot{r}}{N}S^{rt} = -\left[(r^2+n^2)\sin^2\theta\,\dot{\varphi} + 4n\sin^2\left(\frac{\theta}{2}\right)q\right]S^{\varphi t} \tag{66c}$$

The condition $Q = 0$ modifies drastically the form of the solutions.

In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor,

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2} \tag{67}$$

From (52) we can deduce that

$$q = \mathcal{J}^{(0)} + 2nN\sin\theta\frac{C^{\theta\varphi}}{r^2+n^2} \tag{68}$$

$$\dot{\varphi} = \frac{1}{r^2+n^2}\left[\frac{2n\mathcal{J}^{(0)}}{\cos\theta} + \frac{4n^2N}{r^2+n^2}\cdot\frac{4-\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}C^{\theta\varphi} + \frac{\sin\theta}{\cos\theta}C^{\theta\varphi}\right] \tag{69}$$

These relations may be integrated to obtain the expressions for φ and t. We can deduce $\dot{r}$ from the energy, given in (56).

We now study the special case of motion in a plane, for which we choose $\theta = \pi/2$. For scalar particles any solution would actually describe planar motion, because the orbital angular momentum of a scalar particle is always conserved. But this is no longer true in general for spinning particles, for which only the total angular momentum is constants of motion. Planar motion of spinning particles is strictly possible only in special cases, in which orbital and spin angular momentum are separately conserved. This happens only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel (Rietdijk and van Holten, 1993).

For $\theta = \pi/2$ the equations of motion are

$$\dot{S}^{r\theta} = -\frac{r\dot{r}}{r^2+n^2}S^{r\theta} - \frac{2n^2N}{r^2+n^2}\dot{\varphi}S^{r\varphi} - [rN - (r^2+n^2)V]\,\dot{\varphi}S^{\theta\varphi} \tag{70a}$$

$$\dot{S}^{r\varphi} = -\frac{r\dot{r}}{r^2+n^2}S^{r\varphi} \tag{70b}$$

$$\dot{S}^{\theta\varphi} = -\frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} + \frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta} \tag{70c}$$

$$\dot{S}^{\theta t} = \left(\frac{V}{N} - \frac{r}{r^2+n^2}\right)\dot{r}S^{\theta t} - \frac{2n^2N}{r^2+n^2}\dot{\varphi}S^{\varphi t} - 2n\left(\frac{r}{r^2+n^2} - \frac{2V}{N}\right)\dot{\varphi}S^{r\theta}$$

$$+ 2n\left(\frac{r}{r^2+n^2} - \frac{V}{N}\right)\dot{r}S^{\theta\varphi} \tag{70d}$$

$$\dot{S}^{rt} = \frac{V}{N}\dot{r}S^{rt} - n\dot{\varphi}S^{r\theta} + 2n\left(\frac{r}{r^2+n^2} - \frac{V}{N}\right)\dot{r}S^{r\varphi} + [rN - (r^2+n^2)V]\dot{\varphi}S^{\varphi t} \tag{70e}$$

$$\dot{S}^{\varphi t} = \left(\frac{V}{N} - \frac{r}{r^2+n^2}\right)\dot{r}S^{\varphi t} + n\dot{\varphi}S^{\theta\varphi} - \frac{r\dot{\varphi}}{r^2+n^2}S^{rt} - 2n\left(\frac{r}{r^2+n^2} - \frac{2V}{N}\right)\dot{\varphi}S^{r\varphi} \tag{70f}$$

Case (i). In this case the solution describes a particle moving along a fixed radius, for which $\dot{\varphi} = 0$. We are able to obtain a simple exact solution,

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{r^2+n^2}} \tag{71a}$$

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2} \tag{71b}$$

$$S^{\theta t} = \left(\frac{N}{r^2+n^2}\right)^{1/2} C^{\theta t} - \frac{nC^{\theta\varphi}}{r^2+n^2} \tag{71c}$$

$$S^{rt} = \sqrt{N}C^{rt} - 2n\frac{C^{r\varphi}}{\sqrt{r^2+n^2}} \tag{71d}$$

$$S^{\varphi t} = \left(\frac{N}{r^2+n^2}\right)^{1/2} C^{\varphi t} \tag{71e}$$

A special interest represents the case when the supersymmetry constraint $Q = 0$. From this condition we obtain,

$$\frac{\dot{r}}{N}S^{r\theta} = (r^2+n^2)\dot{\varphi}S^{\theta\varphi}, \qquad \frac{\dot{r}}{N}S^{rt} = -(r^2+n^2)\dot{\varphi}S^{\varphi t} \tag{72}$$

For $\dot{\varphi} = 0$ we only have a spin component nenule,

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2} \tag{73}$$

In this case $\dot{r}$ and $\dot{t}$ have a simple expression,

$$\dot{r} = \sqrt{2EN}, \qquad \dot{t} = \left[\frac{4n^2N}{r^2+n^2} - 1\right]\frac{\mathcal{J}^{(0)}}{N} \tag{74}$$

Case (ii). This possibility concerns motion for which $\dot{\varphi} \neq 0$. From $Q = 0$ we obtain the following relations:

$$\frac{\dot{r}}{N}S^{r\theta} = -\mathcal{J}^{(3)}S^{\theta\varphi}, \qquad \frac{\dot{r}}{N}S^{rt} = -\mathcal{J}^{(3)}S^{t\varphi}. \tag{75}$$

It is very interesting that even in this case we have a spin component nenule:

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 + n^2} \tag{76}$$

In this case the expressions for the $\dot{t}$, $\dot{\varphi}$, and $\dot{r}$ can be integrated to give the full solution of the equations of motion for all coordinates and spins:

$$\dot{t} = -\frac{\mathcal{J}^{(0)}}{N} - 2n\left(\frac{C^{\theta\varphi}}{r^2 + n^2} + \dot{\varphi}\right) \tag{77a}$$

$$\dot{r} = \{N(2E - (r^2 + n^2)\dot{\varphi}^2)\}^{1/2} \tag{77b}$$

$$\dot{\varphi} = \frac{1}{r^2 + n^2}\left(\mathcal{J}^{(3)} + 6n^2 N \frac{C^{\theta\varphi}}{r^2 + n^2}\right). \tag{77c}$$

5. NONGENERIC SYMMETRY

In this section we apply the results of section 2 to investigate a new type of supersymmetry in the NUT–RN space–time described by the metric in (43). The electromagnetic field in this space–time is described by

$$F = e(r^2 + n^2)^{-4}(r^2 - n^2)dr \wedge \left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\varphi\right] - 2e(r^2 + n^2)^{-4} nr \sin\theta d\theta \wedge r^2 d\varphi \tag{78}$$

The Killing–Yano tensor is obtained from

$$\frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu = -n\, dr \wedge \left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\varphi\right] - r\sin\theta d\theta \wedge r^2 d\varphi \tag{79}$$

The vielbein $e^a_\mu(x)$ has the following expressions:

$$e^0_\mu\, dx^\mu = -\sqrt{N}\left[dt + 4n\sin^2\left(\frac{\theta}{2}\right)d\varphi\right]$$

$$e^1_\mu\, dx^\mu = \frac{1}{\sqrt{N}}\, dr$$

$$e^2_\mu\, dx^\mu = \sqrt{(r^2 + n^2)}\, d\theta$$

$$e^3_\mu\, dx^\mu = -\frac{\sin\theta}{\sqrt{(r^2 + n^2)}} r^2 d\varphi \tag{80}$$

The components of $f^a_\mu(x)$ can be written as follows:

$$
\begin{aligned}
f^0_\mu\, dx^\mu &= -\frac{n}{\sqrt{N}}\, dr \\
f^1_\mu\, dx^\mu &= \sqrt{N} n \left[dt + 4n \sin^2\left(\frac{\theta}{2}\right) d\varphi \right] \\
f^2_\mu\, dx^\mu &= -\frac{r^3 \sin\theta}{\sqrt{(r^2+n^2)}}\, d\varphi \\
f^3_\mu\, dx^\mu &= r\sqrt{(r^2+n^2)}\, d\theta
\end{aligned}
\tag{81}
$$

Using Eq. (29) we get the components of $C_{abc}(x)$ as follows:

$$
C_{012} = 0, \qquad C_{013} = 0, \qquad C_{023} = 0, \qquad C_{123} = -2\sqrt{N} \tag{82}
$$

Inserting the quantities derived in Eqs. (81) and (82) into Eq. (30) we obtain the new supersymmetry generator Q_f for the NUT–RN space–time. From Eq. (33)–(35) we can construct the Killing tensor, vector, and scalar. The results are

$$
\begin{aligned}
K_{\mu\nu}(x)\, dx^\mu dx^\nu &= -\frac{n^2}{N}\, dr^2 + n^2 N \left[dt + 4n \sin^2\left(\frac{\theta}{2}\right) d\varphi \right]^2 \\
&\quad + \frac{r^6 \sin^2\theta}{r^2+n^2}\, d\varphi^2 + r^2(r^2+n^2)\, d\theta^2
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
I_\mu(x)\, dx^\mu &= 2r^2\sqrt{N} \left(\frac{r \sin\theta}{\sqrt{(r^2+n^2)}} S^{r\varphi} + \sqrt{N} \cos\theta\, S^{\theta\varphi} \right) d\varphi \\
&\quad - (r^2+n^2) N \cos\theta S^{\theta\varphi}\, d\varphi \\
&\quad - \sqrt{((r^2+n^2)N)}(r \sin\theta S^{r\varphi} + \sqrt{((r^2+n^2)N)} \cos\theta S^{\theta\varphi})\, d\varphi \\
&\quad + \sqrt{((r^2+n^2)N)}(n S^{t\varphi} + r S^{r\theta})\, d\theta
\end{aligned}
\tag{84}
$$

$$
G = \frac{-2en}{r^2+n^2} S^{tr} S^{\theta\varphi} \tag{85}
$$

From the Poission–Dirac brackets (15) it can be verfied in a straight forward manner that these equations satisfy the $S0(3, 1)$ algebra.

The expression for Q_f and Eqs. (83)–(85) then define the conserved charge $Z = i/2\{Q_f, Q_f\}$.

6. REMARKS

Our main concern has been the motion of pseudo-classical spinning particles in the NUT–RN space–time. In this analysis we have restricted ourselves to the contribution of the spin contained in the Killing scalars $B^{(\alpha)}(x, \psi)$ defined by (49). In spite of the complexity of the equations, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetry constraint $Q = 0$ plays a very important role for the forms of solutions.

The supersymmertric extension of the NUT–RN space–time admits four standard supersymmetries along with several fermionic symmerties. The existence of such fermionic symmetries is closely related to the existence of Killing–Yano tensors (Vaman and Visinescu, 1996; Gibbons *et al.*, 1993).

The Killing tensor $K_{\mu\nu}$ given in (83) defines a constant of motion (directly) for spinless particles in NUT–RN space–time, whereas for spinning particles it requires the nontrivial contributions from spin, which involve the Killing vector and Killing scalar computed in (84) and (85). This spin-dependent part is described by the antisymmetric Killing–Yano tensor $f_{\mu\nu}$, which satisfies (23) and is the square root of the Killing tensor.

The results of this paper may be interesting in the study of fermion modes in gravitational instantons as well as in long-range monopole dynamics. The results can be specialized for the NUT, Reissner–Nordstrom, and Schwarzschild space–times for $e = 0$, $n = 0$, and $e = n = 0$, respectively. The results can be extended to the NUT–RN space–time generalized with a cosmological parameter. This type of extension may be interesting from the point of view of an inflationary scenario of early universe.

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